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SELF-INTERSECTION NUMBERS AND RANDOM SURFACES ON THE LATTICE

P. Teotonio-Sobrinho *

Department of Physics, University of Illinois at Chicago,
Chicago, IL 60607-7059, USA.

Abstract

String theory in 4 dimensions has the unique feature that a topological term, the oriented self-intersection number, can be added to the usual action. It has been suggested that the corresponding theory of random surfaces wold be free from the problem encountered in the scaling of the string tension. Unfortunately, in the usual dynamical triangulation it is not clear how to write such a term. We show that for random surfaces on a hypercubic lattice however, the analogue of the oriented self-intersection number $I[\sigma]$ can be defined and computed in a straightforward way. Furthermore, $I[\sigma]$ has a genuine topological meaning in the sense that it is invariant under the discrete analogue of continuous deformations. The resulting random surface model is no longer free and may lead to a non trivial continuum limit.

*e-mail: teotonio@tigger.cc.uic.edu

1 Introduction

String theory is supposed to play a crucial role in our understanding of fundamental physics. Either as an effective model for strong interactions or as a fundamental theory for unification, it has been the focus of numerous investigations. In its Euclidean version, string theory is the statistical mechanics of random surfaces. A 2d manifold M is immersed on a target space X resulting on a surface $s \subset X$. The partition function is given by a weighted integral over all surfaces s. Theories of random surfaces are naturally related to many different physical systems such as membranes in biophysics [1] and 3d Ising model [2, 3]. From this perspective, there is enough motivation to investigate lattice versions of random surfaces and study their continuum limit. From the QCD point of view, it would be very interesting to have a lattice version of 4-dimensional strings with a nontrivial continuum limit. Unfortunately, such goal seems to be very difficult.

In the past few years much work has been done in this direction. The simplest and most natural model is the immediate translation of the Nambu-Goto theory. One starts by replacing X by a $\mathbb{Z}^d$ lattice and surfaces by polyhedra made of 2 dimensional plaquettes on $\mathbb{Z}^d$. The partition function is then defined to be the sum over all surfaces σ with a statistical weight given by the number of plaquettes of σ. If M is assumed to have the topology of a sphere, the model is called planar random surfaces [4]-[7]. Such a model is directly related to the $1/n$ limit of $SU(n)$ lattice Yang-Mills theory. Surprisingly, the planar random surface model was proved to be trivial [7] and it can not describe any QCD physics.

Another approach, called dynamical triangulation [12], is used to discretize Polyakov's string theory. The base manifold M is replaced by a generic triangulation where the lengths of the links are taken to be equal. The embedding of a given triangulation on a continuous manifold X defines a surface s and the action (Gaussian) can be taken to be the area of s. In addition to the sum over immersions, one also sums over all possible triangulations in order to take into account the intrinsic geometry of M. The important question is whether the model has a well defined continuum limit. It has been shown that the string tension does not tend to zero at the critical point, giving rise to pathologically crumpled surfaces. Consequently this simple model does not lead to a sensible continuum limit [8]. A natural attempt to overcome the problem is to add to the action a term depending on the extrinsic curvature of s in order to suppress the contributions of "spiked" surfaces. Analytic calculations, using Nambu-Goto action plus an extrinsic curvature term, suggest that the corresponding coupling constant renormalizes to zero and consequently the discrete action can not have a nontrivial continuum limit [9]-[11]. On the other hand much numerical work has been carried out to simulate Polyakov's action together with extrinsic curvature terms [13]. Some evidence of scaling has been found.

Adding an extrinsic curvature term is not the only way of modifying the usual Gaussian theory. When the target space X is 4-dimensional, a new possibility is available for string theory. It has been shown that a topological term can be added to the usual string action

[14] (See also [10, 15]). It introduces an extra weight factor given by $exp(i\theta I[\sigma])$, where $I[s] \in \mathbb{Z}$ is a topological number. In a sense, it is the analogue of the θ-term in QCD. The integer number $I[s]$, the so-called oriented self-intersection number, is a measure of how the embedding of M on X self-intersects. The resulting theory is described by the partition function

$$Z(\lambda, \theta) = \int \mathcal{D}s \; e^{-\lambda A[s] + i\theta I[s]}, \tag{1.1}$$

where $A[s]$ is the usual Nambu-Goto term given by the area of s. It has been suggested [10] that such a partition function would describe smooth surfaces for $\theta = \pi$. Therefore it would be a better candidate for an effective theory of QCD. The presence of the analogue of a θ-term is a very suggestive indication [15].

One way of studying (1.1) is to introduce a lattice regularization and make the functional integral into a sum. Unfortunately it is not so clear how to proceed due to the presence of a topological term. This is a problem common to many theories involving topological terms. The first difficulty is to define and compute the corresponding counterparts on the lattice. Secondly, the topological meaning of such terms in a discrete setting is not always clear. A good example of the situation is given by the QCD instanton number on the lattice. We refer to [16] for the discussion of one possible solution to the problem of instanton number.

We would like to have a scheme of discretization for the random surface problem where the definition of oriented self-intersection number $I[s]$ naturally corresponds to the continuum counterpart. If the base space M is discretized, as it it is in dynamical triangulation, it seems to be very difficult, or even impossible, to come out with a discrete counterpart for topological numbers. The reason being that the usual way of discretizing the manifold M is by looking at lattices, i.e. a cell decomposition $K(M)$ made of vertices, links and faces. However, the scalar field describing the string is defined only for the set $K_0(M)$ of all vertices. Consequently, the set of all field configuration is just $\Gamma = X^{n(K_0(M))}$, where $n(K_0(M))$ is the number of vertices. The quantity $I[s]$ is a function on Γ, but unfortunately Γ has no information about M. The situation is clearly not satisfactory and one should try something different.

Some time ago, an alternative approach to discretization was formulated by Sorkin [17]. In this scheme, M is substituted by a finite topological space $Q(M)$ that has the ability of reproducing important topological features of M. When the number of points in $Q(M)$ increases, $Q(M)$ approximates M better and better. It is possible to define a certain continuum limit, where M can be recovered exactly. Subsequent research developed this methods and made then usable for doing approximations in quantum physics [18]. It turns out that this techniques can be nicely applied to self-intersecting surfaces. Indeed, as it will be explained, the corresponding definition of $I[s]$ for the discrete theory is a faithful translation of the definition in the continuum. Furthermore, it has a truly topological meaning.

We learned from previous work that there are two alternatives for discretization in terms of finite topological spaces. In the first approach, both the base space M and the

target space X are replaced by the discrete spaces $Q(M)$ and $Q(X)$. In the second one, only the target space X is discretized. The second possibility is not very useful for a generic field theory, however it can be efficiently applied to string theory. In this paper we adopt the second possibility with additional restrictions on $Q(X)$. Under these specific conditions the resulting formalism can be reinterpreted in terms of random surfaces made of plaquettes embedded on a usual hypercubic lattice. Although this work was inspired by looking at finite topological spaces, they will not be explicitly mentioned here. An account of self-intersection numbers when both target and base spaces are discretized will be reported elsewhere.

In this paper, we present a discrete model for random surfaces corresponding to the Nambu-Goto theory modified by the presence of the topological term. The case of surfaces with no handles is a modification of the usual planar surfaces. We argue that the pathological behavior observed for the Gaussian action, i.e., $\theta = 0$ in (1.1), may not occur for other values of θ, possibly leading to a nontrivial continuum limit.

The discretization of (1.1), without the term $exp(i\theta I[\sigma])$, has been extensively studied in the past [4]-[7]. Our main objective in this paper is to include the topological term, or in other words, to make sense of the self-intersection number $I[\sigma]$ for any configuration in the model. We will show that $I[\sigma]$ has all the properties that we want. It is an integer, gives the right answer in the continuum limit, and it has a topological meaning in the sense that it is invariant under the analogue of continuous deformations of σ.

To make the paper self contained, the usual self-intersection number for the continuum case is reviewed in Section 2. For the same purpose, some elements of the theory of cell complexes and homology are briefly mentioned in Section 3. The discrete model is discussed in Section 4. The self-intersection number is first defined for a very special class of configurations. Finally, the extension of $I[\sigma]$ for an arbitrary configuration is given by an explicit formula. The topological invariance of $I[\sigma]$ is also demonstrated. Some generic comments on the consequences of the term $I[\sigma]$ are collected in Section 5.

2 The Usual Intersection Number

Consider a 2d manifold M without boundary (parameter space) and a fixed 4d target manifold X. For simplicity one can take X to be $\mathbb{R}^4$. Let $\varphi : M \to X$ be a continuous map (immersion) and $s \subset X$ the surface determined by φ. Different points of M can be mapped to the same point of X. Therefore the surface s can have self-intersections. The self-intersection number $I[s]$ is a measure of how s self-intersects. Usually $I[s]$ is given in terms of local fields. Let ξ_a,$(a = 1, 2)$ be local coordinates of M and φ^μ,$(\mu = 1, 2, 3, 4)$ the components of φ. Then the integer $I[s]$ is given by [14, 10, 15]

$$I[s] = \frac{-1}{16\pi} \int d^2\xi \sqrt{g} g^{ab} \nabla_a t^{\mu\nu} \nabla_b \tilde{t}^{\mu\nu} \tag{2.1}$$

where

$$g_{ab} = \frac{\partial \varphi^\mu}{\partial \xi^a} \frac{\partial \varphi^\mu}{\partial \xi^b} \,,$$

$$t^{\mu\nu} = \frac{\epsilon^{ab}}{\sqrt{g}} \partial_a \varphi^\mu \partial_b \varphi^\nu$$

and

$$\tilde{t}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} t^{\alpha\beta}.$$

If s and s' are homotopic, i.e. they can be continuously deformed into each other, then $I[s] = I[s']$. We will use the notation $s \sim s'$ to indicate homotopy.

The intuitive notion of self-intersection number is very simple. For a 2d surface in 4 dimensions, self-intersection can happen on regions of dimension two, one and zero. Suppose s self-intersects only at a certain number n of isolated points. Furthermore, assume that at any intersection point the two branches of s are not tangent to each other. In this case we say that s is transversal. The simplest invariant associated with s is $I_2[s]$, or intersection module 2. $I_2[s]$ is zero or 1 if n is respectively even or odd. Given any surface s' with transversal self-intersection, one can show that $I_2[s'] = I_2[s]$ if $s' \sim s$. The invariant $I_2[s]$ is extended to non transversal configurations in the following way. Find a transversal surface $\tilde{s} \sim s$ and define $I_2[s]$ to be equal to $I_2[\tilde{s}]$. This definition is motivated by a theorem stating that $\tilde{s}$ always exists and can be made infinitesimally close to s [19].

In this paper we will make use of a distinct, but equivalent [20], presentation of $I[s]$. It turns out that the invariant $I[s]$ in (2.1) can be seen as a refinement of $I_2[s]$. Instead of simply counting the number of intersections, one associates "charges" ± 1 to each intersection and sums over all charges. Let W be the set of points $x_i \in X$ such that $\varphi(p) = \varphi(p') = x_i$, for some pair $p, p' \in M$. Consider also a positive oriented base $\{v_1, v_2\}$ of tangent vectors at $p \in M$ and similarly for $\{w_1, w_2\}$ at $p' \in M$. The map φ will induce two sets $\{v_1', v_2'\}$ and $\{w_1', w_2'\}$ of vectors tangent to s at x_i. We say that s is transversal iff, for all $x_i \in W$, the sets

$$B(x_i) := \{v_1', v_2', w_1', w_2'\} \tag{2.2}$$

are linearly independent. Therefore, for each x_i, $B(x_i)$ is a base of tangent vectors and defines an orientation, called product orientation at x_i. One can compare the product orientation for each x_i with the pre-existent orientation of X and assign a "charge" $+1$ if the orientations agree, and -1 otherwise. The oriented self-intersection number $I[s]$ is defined to be the sum of all such "charges". Observe that there is a potential ambiguity in (2.2), because we can exchange $\{v_1', v_2'\}$ and $\{w_1', w_2'\}$. Obviously, this is not the case, since it does not affect the orientation of $B(x_i)$.

The definition of transversality presented so far makes use of tangent vectors, and this is a problem when dealing with discrete spaces. Fortunately, there is an alternative way of defining transversality that is more useful for us. In some coordinate system, a small neighborhood of an intersection point x_i can be identified with an open set U_i of $\mathbb{R}^4$, where x_i sits at the origin. Let us call s_i and s_i' the two branches of $s \cap U_i$. Transversality

means that we can find a local coordinate system for U_i such that the points of s have coordinates of the form $(y_1, y_2, 0, 0)$ for s_i and $(0, 0, y_3, y_4)$ for s'_i. In other words, U_i can be identified with the Cartesian product $s_i \times s'_i$. If we give to $s_i \times s'_i$ the product orientation, then

$$U_i = I[s_i, s'_i]\ s_i \times s'_i\,, \tag{2.3}$$

where $I[s_i, s'_i] = \pm 1$. The oriented self-intersection number is defined to be [19]

$$I[s] = \sum_i I[s_i, s'_i] \tag{2.4}$$

Formula (2.4) is valid only for transversal surfaces. The extension of this definition to an arbitrary configuration depends on the result mentioned before. Two homotopic transversal configurations have the same $I[s]$, and for any non-transversal s, there is a transversal $\tilde{s}$ such that $\tilde{s} \sim s$. In the same way as for $I_2[s]$, one can safely define $I[s]$ to be $I[\tilde{s}]$.

The main advantage of (2.4) is that it can be generalized to the discrete situation. However, this approach to self-intersection does not give a way of computing $I[s]$ for non-transversal configurations. In this sense, the integral formula (2.1) is more useful, but unfortunately very difficult to be translated to the lattice. For this reason, we will work with the discrete version of (2.4). Finally, in Section 4.3 we will give an explicit formula to compute the self-intersection number for arbitrary configurations.

3 Hypercubic Lattices

In this section we briefly review some notions of homology theory that we will need. We refer to [21] for a systematic exposition.

Abstractly, an n-cell $\alpha_{(n)}$ is a space (of dimension n), together with subspaces $\alpha^i_{(n-1)} \subset \alpha_{(n)}$ called faces. The subsets $\alpha^i_{(n-1)}$ are themselves $(n-1)$-cells, so we can consider their corresponding $\alpha^j_{(n-2)}$ faces. The kind of cells that we will be interested in are regular, meaning that any $\alpha^j_{(n-2)}$ belongs to exactly two $(n-1)$-cells in $\alpha_{(n)}$. By definition, an 1-cell have only two 0-cells as faces, and a 0-cell has no faces. A cell complex K of dimension n is defined to be an union of n-cells and it is totally characterized by its elements $\alpha^l_{(k)}$ and their inclusion relations. Therefore, two abstract complexes K^1 and K^2 are regarded as identical if there is an one to one map $f : K^1 \to K^2$ that preserves the inclusion relations. It is customary to indicate by $K_{(p)} \subset K$ the union of all cells of dimension p. Concretely, an (regular) n-cell $\alpha_{(n)}$ and the corresponding $\alpha^i_{(n-1)} \subset \alpha_{(n)}$ can be realized as n-dimensional polygon in $\mathbb{R}^n$ and respective $(n-1)$-dimensional faces.

A cell decomposition of a n-manifold Y is an abstract complex $K(Y)$ of dimension n such that its concrete realization is homeomorphic to Y. An important property of $K(Y)$ is that any two $(n-1)$-cells belong to at most two n-cells.

Given two abstract cell complexes K^1 and K^2, one can define the product cell complex $K^1 \times K^2$. The cells of $K^1 \times K^2$ are ordered pairs of cells

$$\alpha^{i,j}_{(n+m)} := \left(\alpha^i_{(m)}, \alpha^j_{(n)}\right), \quad \alpha^i_{(m)} \subset K^1 \quad \alpha^j_{(n)} \subset K^2 \tag{3.1}$$

together with the inclusion relations

$$\left(\alpha^k_{(m-1)}, \alpha^l_{(n-1)}\right) \subset \left(\alpha^i_{(m)}, \alpha^j_{(n)}\right) \quad \text{iff} \quad \alpha^k_{(m-1)} \subset \alpha^i_{(m)} \ \text{ and } \ \alpha^l_{(n-1)} \subset \alpha^j_{(n)}. \tag{3.2}$$

In this paper, we will be restricted to consider n-cells that can be realized as cubes of dimension n. Abstractly, a cubic n-cell $L_{(n)}$ is by definition the product

$$L_{(4)} = L^1 \times L^2 \times ... \times L^n \tag{3.3}$$

of n 1-cells L^i. In other words, a cell $\alpha \subset L_{(n)}$ is given by

$$\alpha = (\alpha^1, \alpha^2, ..., \alpha^n) \tag{3.4}$$

where α^i can be L^i or one of its vertices. A cubic cell complex of dimension n will be the union of cubic cells of dimension n.

Given a cell complex K, one defines the vector space $C_n(K, \mathbb{Z})$ as the linear combination of n-cells, with coefficients in $\mathbb{Z}$

$$C_n(K, \mathbb{Z}) = \left\{ \xi_{(n)} = \sum_i \lambda_i \alpha^i_{(n)} \ : \ \lambda_i \in \mathbb{Z}, \quad \alpha^i_{(n)} \subset K \right\} \tag{3.5}$$

The vectors $\xi_{(n)}$ are called n-chains. The direct sum of all $C_n(K, \mathbb{Z})$ will be denoted by $C(K, \mathbb{Z})$.

The definition of orientation is related to a linear operator

$$\partial : C_n(K, \mathbb{Z}) \to C_{(n-1)}(K, \mathbb{Z}),$$

called the boundary operator. It is enough to define ∂ for the base elements $\alpha^i_{(n)}$. Intuitively, the boundary $\partial\alpha^i_{(n)}$ of an n-cell $\alpha^i_{(n)}$ has to do with its faces. In other words, it is a linear combination, with coefficients ± 1, of all $(n-1)$-cells $\alpha^j_{(n-1)}$ such that $\alpha^j_{(n-1)} \subset \alpha^i_{(n)}$. We define

$$\partial\alpha^i_{(n)} = 0 \quad \text{if } n = 0 \tag{3.6}$$

and

$$\partial\alpha^i_{(n)} = \sum_j I_{nc}(\alpha^i_{(n)}, \alpha^j_{(n-1)})\alpha^j_{(n-1)}. \tag{3.7}$$

The coefficients $I_{nc}(\alpha^i_{(n)}, \alpha^j_{(n-1)}) = \pm 1$ are called the incidence numbers and they have to be assigned in such way that

$$\partial\partial\xi = 0 \qquad \text{for any } \xi \in C(K, \mathbb{Z}). \tag{3.8}$$

In terms of incidence numbers, (3.8) is equivalent to

$$\sum_j I_{nc}(\alpha^i_{(n)}, \alpha^j_{(n-1)}) I_{nc}(\alpha^j_{(n-1)}, \alpha^k_{(n-2)}) = 0. \tag{3.9}$$

It turns out that incidence numbers can be assigned recursively in a simple way, and this is what is used to define orientation. First, it is assumed that the boundary of an 1-cell $\alpha_{(1)}$ with faces $\alpha^1_{(0)}$ and $\alpha^2_{(0)}$ can only be $\pm(\alpha^2_{(0)} - \alpha^1_{(0)})$. In other words, for a given i there are only two possibilities for $I_{nc}(\alpha^i_{(1)}, \alpha^j_{(0)})$, and one is the negative of the other. Suppose now that all $I_{nc}(\alpha^i_{(1)}, \alpha^j_{(0)})$ have been chosen for a given 2-cell $\alpha^k_{(2)}$. It is easy to see that there are only two possibilities for $I_{nc}(\alpha^k_{(2)}, \alpha^i_{(1)})$ satisfying (3.9) and one is the negative of the other. This is actually a general fact. Once the incidence numbers are chosen for the faces $\alpha^i_{(n-1)}$ of an n-cell $\alpha^k_{(n)}$, there are only two possible choices for $I_{nc}(\alpha^k_{(n)}, \alpha^i_{(n-1)})$.

From above it follows that, once we find a possible configuration of incidence numbers, all the others can be obtained by a certain set of transformations. Let us introduce a function $g(\alpha)$ from $K(Y)$ to $\{-1, +1\}$. Given a possible configuration of incidence numbers I^0_{nc} we define a new configuration gI^0_{nc}

$$gI^0_{nc}(\alpha^i_{(r)}, \alpha^j_{(r-1)}) = g(\alpha^i_{(r)})\ I^0_{nc}(\alpha^i_{(r)}, \alpha^j_{(r-1)})\ g(\alpha^j_{(r-1)}). \tag{3.10}$$

It is clear that the gI^0_{nc} satisfy (3.9). Furthermore, all possibilities for I_{nc} can be generated in this way, by starting from any I^0_{nc}.

Now consider an n-dimensional complex $K(Y)$ associated with some n-manifold Y. What we call local orientations of K is the freedom to choose independently $I_{nc}(\alpha^i_{(n)}, \alpha^j_{(n-1)})$ at each n-cell. We say that two configurations I_{nc} and I'_{nc} define the same local orientation of $K(Y)$, or are equivalent $I_{nc} \sim I'_{nc}$, if they are related by a transformation (3.10) with $g(\alpha^i_{(n)}) = 1$

$$I_{nc} \sim I'_{nc} \quad \text{iff} \quad I'_{nc} = gI_{nc} \quad \text{for some } g \text{ such that } g(\alpha^i_{(n)}) = 1.$$

A global orientation for $K(Y)$ appears when we start to compare the local orientations for neighboring n-cells. We say that the local orientation at $\alpha^1_{(n)}$ agrees with the local orientation at $\alpha^2_{(n)}$ iff

$$I_{nc}(\alpha^1_{(n)}, f_{(n-1)}) = -I_{nc}(\alpha^2_{(n)}, f_{(n-1)}), \tag{3.11}$$

where $f_{(n-1)}$ is the unique common face. An n-dimensional complex, together with an orientation, is called oriented iff all the local orientations agree.

A vector $\xi_{(n)} \in C_{(n)}(K, \mathbb{Z})$ of the form

$$\xi_{(n)} = \sum_i s_i \alpha^i_{(n)}\ ; \qquad s_i = \pm 1 \tag{3.12}$$

is interpreted as the oriented n-dimensional subcomplex of K given by

$$\bigcup_i s_i \alpha^i_{(n)},$$

where the factors $s_i = \pm 1$ indicate orientation. If $\xi_{(n)}$ is globally oriented, then $\partial\xi_{(n)}$ is also globally oriented.

Let K^1 and K^2 be two globally oriented cell complexes and ∂_1 and ∂_2 be boundary operators defined on $C(K^1, \mathbb{Z})$ and $C(K^2, \mathbb{Z})$. Consider an operator ∂ acting on $C(K^1 \times K^2, \mathbb{Z})$ in the following way

$$\partial(\alpha^1_{(m)}, \alpha^2_{(n)}) = (\partial_1 \alpha^1_{(m)}, \alpha^2_{(n)}) + (-1)^m (\alpha^1_{(m)}, \partial_2 \alpha^2_{(n)}) \,, \tag{3.13}$$

It follows immediately that $\partial^2 = 0$. Therefore I^1_{nc} and I^2_{nc} will define a configuration of incidence numbers $I^1_{nc} \times I^2_{nc}$ for $K^1 \times K^2$. If $g_1 I^1_{nc}$ and $g_2 I^2_{nc}$ are two other incidence numbers equivalent to I^1_{nc} and I^2_{nc}, a simple calculation shows that

$$g_1 I^1_{nc} \times g_2 I^2_{nc} = g_{(1\times 2)} I^1_{nc} \times I^2_{nc} \tag{3.14}$$

where $g_{(1\times 2)}\left((\alpha^1, \alpha^2)\right) = g_1(\alpha^1) g_2(\alpha^2)$. Therefore (3.13) induces a canonical orientation on $K^1 \times K^2$, called the product orientation. It is a simple exercise to verify that the product orientation is also global.

An 1-cell, or link L^i, is totally determined by its vertices a^i and b^i. It is standard to write $[a^i, b^i]$ for L^i and define

$$\partial[a^i, b^i] = [a^i] - [b^i] \tag{3.15}$$

then $[b^i, a^i]$ will be identified with $-L^i$. Therefore, the cube

$$L_{(4)} = ([a^1, b^1], [a^2, b^2], [a^3, b^3], [a^4, b^4]) \tag{3.16}$$

has a standard set of incidence number determined by (3.15) and (3.13). Whenever we write a cubic cell as in (3.16), the standard incidence numbers are assumed.

4 The Discrete Model

The discretization is done by introducing a grid on the space Γ of all surfaces. This allow us to write the functional integral (1.1) as a sum. In other words, Γ will be substituted by some discrete space Γ_d, where we can define an area $A[\sigma]$ and and an intersection number $I[\sigma]$ for any configuration $\sigma \in \Gamma_d$.

4.1 Space of Configurations

The space of configurations we need to consider is given by the set of all immersions φ of a 2-dimensional manifold M on some 4-dimensional target space X.* However, the action

*We assume that X has no boundary, but M may have boundary components.

(Nambu-Goto) depends only on the area of the surface s determined by φ. Any two immersions that give the same surface s in X are regarded as equivalent. The relevant set of configurations Γ is then the set of all such surfaces. Evidently not all s are submanifolds of X. They can be degenerated surfaces in the sense that they can fold on themselves, i.e., more than one point of M can be mapped to the same point of X.

Let us assume, for simplicity, that $X = \mathbb{R}^4$. Consider the discrete lattice $\mathbb{Z}^4$ of points (vertices) $v_i \in X$ that have integer coordinates in some lattice spacing unit a. It determines a cell decomposition $K(X)$ of X where $K_{(0)} = \mathbb{Z}^4$ and $K_{(n)}$ is the set of all n-dimensional elementary cubes determined by $\mathbb{Z}^4$.

It is useful to think of $K(X)$ as the product of 1-dimensional complexes. According to the notation introduced at the end of Section 3, an arbitrary cell $\alpha \subset K(X)$ will be written as

$$\alpha = (\alpha^1, \alpha^2, \alpha^3, \alpha^4) \tag{4.1}$$

where the variable α^i can take the values $[p^i]$ (0-cell) or $[p^j, p^j + 1]$ (1-cell), $p_i \in \mathbb{Z}$. They will be the base elements for $C(K(X), \mathbb{Z})$. Similarly one can also discretize the 4 dimensional torus $\mathbf{T}^4$ by taking $K(X)^{(0)} = (\mathbb{Z}_n)^4$. In this paper we will be limited to examine only these two cases.

We are now in a position to define the discrete space Γ_d that will be used to approximate the infinite dimensional space of configurations Γ. The set Γ_d will be a countable sub set of Γ. A configuration σ belongs to Γ_d if the corresponding surface lies entirely on plaquettes of $K(X)$. Since the number of plaquettes is countable, so is the number of elements in Γ_d.

Another way of interpreting Γ_d is to think of a configuration σ as the natural two dimensional generalization of a random walk. Let us explain. Consider a random walker that starts at a vertex v^0 and then moves to a neighboring vertex v^1. The trajectory, or curve traversed by the random walker, can be specified by the oriented link $[v^0, v^1]$. The subsequent steps can be described by adding more links to one end of the curve. Eventually, the random walker may go to a vertex v that has been visited before, and the curve self-intersects. In this case, the curve is no longer regular, in the sense that v belongs to more than 2 links. Analogously, one starts to construct a surface by marking some 2d subcomplex $K^p \subset K(X)$, where K^p is the union of p plaquettes of $K(X)$. The surface K^p is supposed to be regular in the sense that all 1-cells belongs to at most 2 plaquettes. Alternatively, K^p can be also written as a vector in $C_{(2)}(K(X), \mathbb{Z})$

$$K^p = s_1\alpha^1_{(2)} + s_2\alpha^2_{(2)} + ... + s_p\alpha^p_{(2)}, \quad s_i = \pm 1, \tag{4.2}$$

where $\alpha^i_{(2)}$ are base elements of the form (4.1). Now, one tries to add one more plaquette $s_{p+1}\alpha^{p+1}_{(2)}$ in such way that it has at least one common link with K^p. Eventually, it may happen that, for the resulting complex, some links belong now to more than 2 plaquettes and the resulting complex $K^{p+1} = K^p + s_{p+1}\alpha^{p+1}_{(2)}$ would not be regular. The extreme case is when

$$\alpha^{p+1}_{(2)} = \alpha^j_{(2)}, \quad \text{for some } \alpha^j_{(2)} \text{ in } K^p, \tag{4.3}$$

meaning that $\alpha_{(2)}^{p+1}$ has been previously marked. The idea is to make K^{p+1} regular by hand. One enlarges $K(X)$ by introducing a copy of the base element $\alpha_{(2)}^{p+1}$ and denoting it by $\overline{\alpha_{(2)}^{p+1}}$. Then, K^{p+1} is defined to be the abstract complex given by

$$K^{p+1} = K^p + s_{p+1}\overline{\alpha_{(2)}^{p+1}}. \tag{4.4}$$

and it is regular by construction. The process is iterated a number of times. Eventually, it will be necessary to introduce many copies of a given element $\alpha_{(2)}^i$. They will be denote by $\overline{\alpha_{(2)}^i}$, $\overline{\overline{\alpha_{(2)}^i}}$, etc.

A configuration σ with area n is any abstract cell complex $\sigma = K^n$ constructed as above such that σ represents a cell decomposition of M. We can write σ in the form

$$\sigma = \sum_{i=1}^{l} \beta^i, \tag{4.5}$$

where

$$\beta^i = s_i\alpha_{(2)}^i + \overline{s_i}\,\overline{\alpha_{(2)}^i} + \overline{\overline{s_i}}\,\overline{\overline{\alpha_{(2)}^i}} + \ldots .$$

Notice that a configuration σ can not in general be interpreted as a subcomplex of $K(X)$, i.e., a vector of the form (3.12). This happens only if σ self-intersects on a subcomplex of dimension zero.

There is a very useful map ξ from Γ_d to $C_{(2)}(K(X), \mathbb{Z})$. If σ is as in (4.5), then $\xi(\sigma)$ is defined to be the following vector in $C_{(2)}(K(X), \mathbb{Z})$

$$\xi(\sigma) = (s_1 + \overline{s_1} + \overline{\overline{s_1}} + \ldots)\,\alpha_{(2)}^1 + (s_2 + \overline{s_2} + \overline{\overline{s_2}} + \ldots)\,\alpha_{(2)}^2 + \ldots + (s_l + \overline{s_l} + \overline{\overline{s_l}} + \ldots)\,\alpha_{(2)}^l. \tag{4.6}$$

In other words, all occurrences of $\overline{\alpha_{(2)}^i}$, $\overline{\overline{\alpha_{(2)}^i}}$, etc, in (4.5) are replaced by $\alpha_{(2)}^i$. This map will be used on Section 4.3.

Let $\gamma_i, (i = 1, ..., n)$ denote fixed loops on $K(X)$. The relevant observables are the n point Green functions

$$Z_{n,m}(\gamma_1, ..., \gamma_n; \lambda, \theta) = \sum_{\sigma \in \Gamma_d(\gamma_1, ..., \gamma_n)} e^{-\lambda A[\sigma] + i\theta I[\sigma]} \tag{4.7}$$

The sum is done over the set $\Gamma_d(\gamma_1, ..., \gamma_n)$ of all surfaces with n holes and m handles, such that

$$\partial\sigma = \bigcup_{i=1}^{n} \gamma_i. \tag{4.8}$$

Some correlation functions play a special rule in the analysis of the theory. For example the string tension, is defined by

$$\tau(\lambda, \theta) = \lim_{\gamma \to \infty} \frac{1}{LM} \log Z_{1,m}(\gamma_{LM}; \lambda, \theta), \tag{4.9}$$

where γ_{LM} is a rectangular loop with $L \times M$ links.

The simplest random surface model would be given by the sum over surfaces with fixed topology. Let us assume, for example, $X = \mathbb{R}^4$ and surfaces with no handles. This gives us a generalization of the planar random surface model.

4.2 Intersection Number

As defined on Section 2, the self-intersection number involves the notion of transversality. We would like to have a definition of transversality for our discrete surfaces that is a natural generalization of the definition for continuous surfaces. Let us consider on $\mathbb{R}^{(n+m)}$ two hyper surfaces $s^1_{(m)}$ and $s^2_{(n)}$ of dimensions m and n. Suppose they meet at a point x. We say that they are perpendicular if their tangent vectors are perpendicular. It is also equivalent to say that for a small neighborhood U_x of x, the surfaces are flat and U_x can be canonically identified with $s^1_{(m)} \times s^2_{(m)}$. If this is the case, $s^1_{(m)}$ and $s^2_{(n)}$ are surely transversal. If we are dealing with cubic cells this seems to be the natural notion of transversality. Let us make the idea more precise.

We will use the convention that α^i, v^i and l^i are variables related to the 1-dimensional cell $[p^i, p^i + 1]$, $p^i \in \mathbb{Z}$, with the following ranges

$$\begin{aligned} \alpha^i &= [p^i], [p^i], [p^i, p^i + 1]; \\ v^i &= [p^i], [p^i + 1]; \\ l^i &= [p^i, p^i + 1]. \end{aligned}$$

Consider a cubic cell $\alpha_{(4)} = (l^1, l^2, l^3, l^4)$ as defined on Section 3. Let V, L denote subcells of $\alpha_{(4)}$. Consider V to be a 3d cube and L a link such that they have one common vertex. For example, one can take

$$\begin{aligned} V &= (l^1, l^2, l^3, v^4) \\ L &= (v^1, v^2, v^3, l^4) \end{aligned} \tag{4.10}$$

sharing the vertex (v^1, v^2, v^3, v^4). From (4.10) we see that V and L are perpendicular in the obvious sense. Arbitrary cells ω_V and ω_L of V and L are of the form

$$\omega_V = (\alpha^1, \alpha^2, \alpha^3, v^4) \quad \text{and} \quad \omega_L = (v^1, v^2, v^3, \alpha^4). \tag{4.11}$$

There is a canonical one to one map f between the product complex $V \times L$ and $\alpha_{(4)}$ given by

$$f\left((\omega_V, \omega_L)\right) = (\alpha^1, \alpha^2, \alpha^3, \alpha^4). \tag{4.12}$$

One can show that f preserves the inclusion relations and that if $V \times L$ is oriented according to (3.13), it also preserves orientation. Consequently $V \times L$ can be identified with $\alpha_{(4)}$. We will write

$$V \times L = \alpha_{(4)}, \tag{4.13}$$

instead of $f(V \times L) = \alpha_{(4)}$ since the identification (4.12) is canonical. The analogous identification of $L \times V$ with $\alpha_{(4)}$ does not preserve orientation, and we write

$$L \times V = -\alpha_{(4)} \tag{4.14}$$

The last two formulas can be generalized in an obvious way. First let us introduce some notation. Consider a cube $\alpha_{(n+m)} = (l^1, ..., l^{n+m})$ of dimension $(n+m)$. We will write a n-dimensional sub-cell belonging to $\alpha_{(n+m)}$ as

$$\alpha_{[a_1 a_2 ... a_n]} = [(l^{a_1}, l^{a_2}, ..., l^{a_n}, v^{a_{n+1}}, ..., v^{a_{n+m}})], \tag{4.15}$$

where indices a_i take values $(1, ..., n+m)$. The square brackets [.] stands for the ordering of the indices $a_1, a_2, ..., a_n$, or the ordered permutation of the objects with indices $a_1, a_2, ..., a_n$. The notation indicates that $\alpha_{[a_1 a_2 ... a_n]}$ have link components only on the directions $a_1, ..., a_n$. In other words, a generic cell in $\alpha_{[a_1 a_2 ... a_n]}$ is of the form

$$\omega_{[a_1 ... a_n]} = [(\alpha^{a_1}, ..., \alpha^{a_n}, v^{a_{n+1}}, ..., v^{a_{n+m}})]\,. \tag{4.16}$$

Suppose now that we have another subcell $\alpha_{[b_1 ... b_m]}$ that shares exactly one vertex with $\alpha_{[a_1 ... a_n]}$. Consequently the two sets of indices $\{a_1 ... a_n\}$ and $\{b_1 ... b_m\}$ cannot have any elements in common. There is a canonical one to one map $f : \alpha_{[a_1 ... a_n]} \times \alpha_{[b_1 ... b_m]} \to \alpha_{(m+n)}$ given by

$$f((\omega_{[a_1 ... a_n]}, \omega_{[b_1 ... b_m]})) = \left[(\alpha^{a_1}, ..., \alpha^{a_n}, \alpha^{b_1}, ..., \alpha^{b_m})\right]. \tag{4.17}$$

It is a simple matter to show that under this canonical identification we can write

$$\alpha_{[a_1 ... a_n]} \times \alpha_{[b_1 ... b_m]} = \epsilon_{[a_1 ... a_n][b_1 ... b_m]} \alpha_{(n+m)} \tag{4.18}$$

where $\epsilon_{i...k}$ is the usual Levi-Civita symbol.

Observe that, for two arbitrary cells $\alpha_{(n)}$ and $\beta_{(m)}$, the product $\alpha_n \times \beta_m$ always makes sense as an abstract complex. However, its canonical identification with an $(n+m)$-cell $\alpha_{(n+m)}$ only makes sense if they belong to $\alpha_{(n+m)}$ and share a single vertex.

Let $K(X)$ be the cubic cell decomposition of $\mathbb{R}^4$ (or $\mathbf{T}^4$) and σ a configuration in Γ_d. The first condition for σ to be considered transversal is that it self-intersects only on vertices. In particular, σ has to be a 2-dimensional subcomplex of $K(X)$. Let $v = ([p^1], [p^2], [p^3], [p^4])$ be one of the vertices where the self intersection occurs. We define a neighborhood U_v of v to be the union of all 4-cells $\alpha^k_{(4)}$ that contain v. An example of of such a cell is $([p^1 - 1, p^1], [p^2 - 1, p^2], [p^3, p^3 + 1], [p^4, p^4 + 1])$. Since we are restricted to cell decompositions of $\mathbb{R}^4$ (or $\mathbf{T}^4$), U_v is the union of 16 4-cells. In other words, considered as a vector in $C_{(4)}(K(X), \mathbb{Z})$, U_v is given by

$$U_v = \sum_{k=1}^{16} \alpha^k_{(4)}, \tag{4.19}$$

Consider the 1-dimensional complex $L^i = [p^i - 1, p^i] \cup [p^i, p^i + 1]$, or

$$L^i = [p^i - 1, p^i] + [p^i, p^i + 1] \tag{4.20}$$

made of 2 adjacent vertices. One can see that neighborhood U_v is the product of 4 such 1-dimensional complexes. In other words,

$$U_v = (L^1, L^2, L^3, L^4). \tag{4.21}$$

Let σ_v and σ'_v be the two components of $\sigma \cap U_v$. We say that the intersection is transversal iff σ and σ' are of the form

$$\begin{aligned} \sigma_v &= s\left[(L^a, L^b, [p^c], [p^d])\right] \\ & \qquad \text{with } [a,b,c,d] = 1234 \text{ and } s, s' = \pm 1. \\ \sigma'_v &= s'\left[(L^c, L^d, [p^a], [p^b])\right] \end{aligned} \tag{4.22}$$

In this case we can write

$$U_v = I[\sigma_v, \sigma'_v]\ \sigma_v \times \sigma'_v\,, \tag{4.23}$$

where the canonical identification is being used. The coefficient $I[\sigma_v, \sigma'_v] = \pm 1$ is called the intersection number at v. From (4.18), (4.22) and (4.23), it follows that

$$I[\sigma_v, \sigma'_v] = ss'\epsilon_{[ab][cd]}. \tag{4.24}$$

Finally, the self-intersection number $I[\sigma]$ is defined to be the sum of all intersection numbers

$$I[\sigma] = \sum_v I[\sigma_v, \sigma'_v]. \tag{4.25}$$

Let s be the continuous surface associated with a transversal σ. The surface s is transverse in the usual sense, therefore $I[s]$ defined by (2.4) can also be computed. It is a very simple exercise involving tangent vectors to show that $I[s] = I[\sigma]$.

4.3 Topological Invariance and Non-transversal Configurations

The definition of the self-intersection number for transversal configurations, presented in Section 4.2, is the exact analogue of the continuous definition on Section 2. To complete the correspondence with the continuous case we need to introduce on Γ_d a notion of continuous deformations, or homotopy of configurations, and show that $I[\sigma]$ is an invariant. We also have to extend the $I[\sigma]$ to non-transversal configurations.

Intuitively, two configurations σ_1 and σ_2 should be considered homotopic iff their continuous counterparts s_1 and s_2 can be deformed into each other by a sequence of small deformations. Let us explain what is meant by a small deformation. Given $\sigma_1 \in \Gamma_d$, consider a small portion D_1 of σ_1 such that D_1 is a 2-dimensional sub-complex in $K(X)$.

Furthermore, D_1 is required to be topologically equivalent to a disk. A small deformation will be a process where D_1 is removed and substituted by another disk $D_2 \subset K(X)$. We say that the resulting surface σ_2 is a continuous deformation of σ_1 iff $(D_2 \cup -D_1)$ is the boundary of a 3d complex $B \subset K(X)$, or

$$D_2 - D_1 = \partial B, \tag{4.26}$$

where B is topologically equivalent to a 3-dimensional ball. It is clear that a minimal deformation happens when B is a cube, D_1 is one of its plaquettes and D_2 the union of the other 5 plaquettes. However, such a minimal deformation is not enough to generate all small deformations, as we will illustrate by an example. Let D_1 be the union of two adjacent plaquettes as in Fig. 1(a). Let us apply to each plaquette of D_1 the minimal deformation described above. The resulting surface (Fig. 1(b)) consists of two cubic boxes, open on the top, placed side by side. It does not correspond to a regular surface. Notice that it has 2 superposed plaquettes that are glued along the top link (see figure). Another alternative is to deform D_1 into D_2' (Fig. 1(c)), a regular surface consisting of single box open on the top. But for consistency, D_2 has to be homotopic to D_2'. It is clear that, to have a complete set of minimal deformations, we need to include another deformation rule. Two superposed plaquettes glued along some of their links can be removed. Obviously, the links they do not share should remain. An example of the second type of elementary deformations is shown in Fig. 1(d).

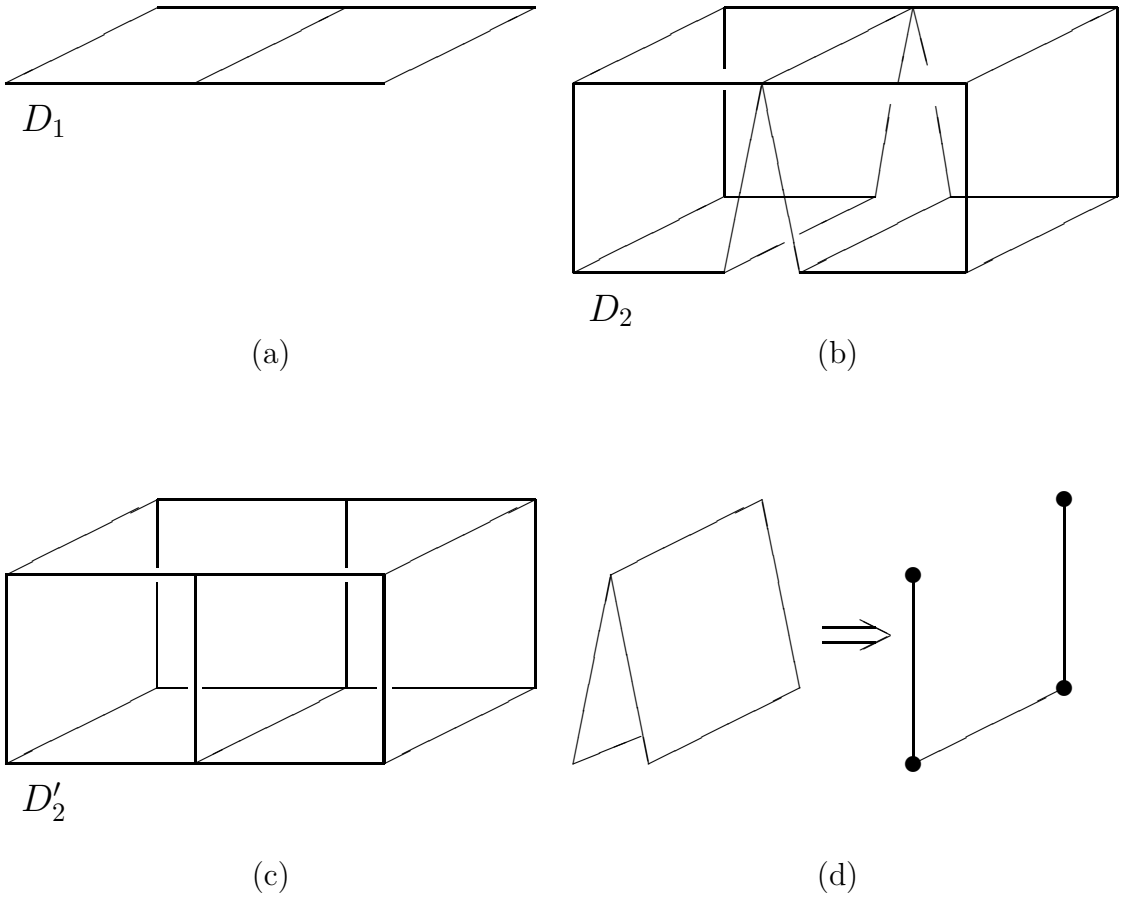

Fig. 1. (a) is the disk D_1 made of two adjacent plaquettes. (b) is a possible continuous transformation D_2, where each plaquette of D_1 is minimally deformed. It consists of two cubic boxes, open on the top, placed side by side. The two superposed plaquettes are drawn slightly separated to make the picture clear. (c) is an alternative deformation D_2' of D_1. The elementary deformation connecting D_2 and D_2' is shown in (d)

We recall that associated to each configuration $\sigma \in \Gamma_d$ there is a 2-chain $\xi(\sigma) \in C_{(2)}(K(X), \mathbb{Z})$ given by (4.6). It turns out that small deformations have a very simple interpretation in terms of chains. If σ_1 and σ_2 differ by a minimal deformation of the type illustrated in Fig. 1(d), it follows from the definitions that $\xi(\sigma_1) - \xi(\sigma_2)$ is equal to zero. In general we have

$$\sigma_1 \sim \sigma_2 \quad \text{implies} \quad \xi(\sigma_2) = \xi(\sigma_2) + \partial B, \tag{4.27}$$

for some 3-chain B.

Equation (4.27) is the clue to the invariance of $I[\sigma]$ under continuous deformations. In order to proceed it will be useful to introduce two kinds of products involving chains. The first one is a scalar product $\langle \cdot, \cdot \rangle$ on $C(K(X), \mathbb{Z})$. As usual it is enough to give the product for the base elements. We define

$$\langle \alpha^i_{(m)}, \alpha^j_{(n)} \rangle = \delta_{mn}\delta_{ij}. \tag{4.28}$$

The second one is a cross product. Let $\alpha^i_{(m)}$ and $\alpha^j_{(n)}$ be base elements such that $(m+n) = 4$. The cross product $\alpha^i_{(m)} \times \alpha^j_{(n)}$ will be given by (4.18) if they belong to the same 4-cell, and will be zero otherwise. For arbitrary chains $\xi^1_{(m)}$ and $\xi^2_{(n)}$ the product is extended by linearity.

Let us regard the cell decomposition $K(X)$ as a vector on $C_{(4)}(K(X), \mathbb{Z})$. We can assume that all 4-cells $\alpha^i_{(4)}$ have a coherent orientation and therefore

$$K(X) = \sum_i \alpha^i_{(4)}, \tag{4.29}$$

where the sum is over all 4-cells. Let σ be a transversal configuration with n plaquettes. In this case, the expression (4.6) reduces to

$$\xi(\sigma) = s_1\alpha^1_{(2)} + s_2\alpha^2_{(2)} + ... + s_n\alpha^n_{(2)}. \tag{4.30}$$

Consider the product $\xi(\sigma) \times \xi(\sigma)$. Because of the way the cross product was defined, most of the n^2 terms in the expansion of $\xi(\sigma) \times \xi(\sigma)$ will be zero. There will be contributions only from plaquettes that share exactly one vertex, or in other words, from plaquettes that contain the intersection points v. It is not difficult to see that there will be 32 non vanishing terms per each intersection point v. From (4.18), (4.22) and (4.24) one can show that each term is equal a 4-cell multiplied by the intersection number $I[\sigma_v, \sigma'_v]$. Combining (4.29) and (4.30) with the previous observation, one can see that the oriented self-intersection number $I[\sigma]$ can be expressed as

$$I[\sigma] = \frac{1}{32}\langle K(X), \xi(\sigma) \times \xi(\sigma) \rangle. \tag{4.31}$$

The topological invariance of (4.31) is a consequence of the identity

$$\langle K(X), \xi_{(2)} \times \partial\xi_{(3)} \rangle = \langle K(X), \partial\xi_{(2)} \times \xi_{(3)} \rangle, \tag{4.32}$$

where $\xi_{(2)}$ and $\xi_{(3)}$ are arbitrary 2-chains and 3-chains. Let us assume (4.32) for the moment. Given $\sigma \sim \sigma'$, it follows from (4.27), (4.31) that

$$I[\sigma'] = \frac{1}{32}\langle K(X), \xi(\sigma) \times \xi(\sigma)\rangle + \frac{1}{16}\langle K(X), \xi(\sigma) \times \partial B\rangle + \frac{1}{32}\langle K(X), \partial B \times \partial B\rangle. \quad (4.33)$$

Using the identities (4.32) and $\partial^2 = 0$ we have

$$I[\sigma'] - I[\sigma] = \frac{1}{16}\langle K(X), \partial\xi(\sigma) \times B\rangle \quad (4.34)$$

If σ has no boundary, then $\partial\xi(\sigma) = 0$ and $I[\sigma] = I[\sigma']$. When the surface σ has a boundary, (4.31) is still well defined, but is no longer invariant under arbitrary deformations. One has to be restricted to the class of deformations such that $\langle K(X), \partial\xi(\sigma) \times B\rangle = 0$. For example, if the fluctuations on σ occur far from its boundary, i.e., $B = 0$ at the boundary of σ, the r.h.s. of (4.34) obviously vanishes.

The extension of $I[\sigma]$ to non-transversal configurations is now obvious. Given σ, one computes $\xi(\sigma)$ by formula (4.6) and uses (4.31) to compute $I[\sigma]$. This is a well-defined procedure, since the r.h.s. of (4.31) makes sense for an arbitrary vector $\xi(\sigma)$ in $C_{(2)}(K(X), \mathbb{Z})$.

We would like to indicate how identity (4.32) can be proven. It is enough to verify it for the base elements in $C_{(2)}(K(X), \mathbb{Z})$ and $C_{(3)}(K(X), \mathbb{Z})$. In the notation of Section 4.2, let

$$\begin{aligned} \xi_{(2)} &= \left[(l^a, l^b, v^c, v^d)\right] \\ \xi_{(3)} &= \left[(\tilde{l}^a, \tilde{l}^i, \tilde{v}^j, \tilde{v}^k)\right]. \end{aligned} \quad (4.35)$$

Notice that $\xi_{(2)}$ and $\xi_{(3)}$ have to have link components on one common direction given by the repeated index a. Let us first compute $\langle K, \partial\xi_{(2)} \times \xi_{(3)}\rangle$. The only terms in $\partial\xi_{(2)}$ that contribute are

$$\epsilon^{ab}\left[([p^a], l^b, v^c, v^d)\right] - \epsilon^{ab}\left[([p^a + 1], l^b, v^c, v^d)\right].$$

After some algebra we have

$$\langle K(X), \partial\xi_{(2)} \times \xi_{(3)}\rangle = \epsilon^{ab}\epsilon^{[ajk]b}\left(\delta_{[p^a+1]\subset\tilde{l}^a} - \delta_{[p^a]\subset\tilde{l}^a}\right), \quad (4.36)$$

where $\delta_{[p^a]\subset\tilde{l}^a}$ is zero unless $[p^a] \subset \tilde{l}^a$. Analogously,

$$\langle K(X), \xi_{(2)} \times \partial\xi_{(3)}\rangle = \epsilon^{a[jk]}\epsilon^{[ab][jk]}\left(\delta_{[\tilde{p}^a]\subset l^a} - \delta_{[\tilde{p}^a+1]\subset l^a}\right). \quad (4.37)$$

The fact that $K(X)$ has no boundary has been used to derive the last two equations. A little thought shows that $\epsilon^{ab}\epsilon^{[ajk]b} = \epsilon^{a[jk]}\epsilon^{[ab][jk]}$. If l^a and $\tilde{l}^a$ do not share any vertex or if $l^a = \tilde{l}^a$, (4.36) and (4.37) are both zero. For the cases where they are adjacent, the r.h.s. of (4.36) and (4.37) give the same result.

5 Final Remarks

A discrete model of random surfaces with topological term was introduced. The model is described by the Green functions $Z_{n,m}(\gamma_1, ..., \gamma_n; \lambda, \theta)$ defined by (4.7). We show that the topological term $I[\sigma]$ is well defined and can be computed explicitly by formula (4.31) for the cases where the target space X is $\mathbb{R}^4$ or the 4-torus $\mathbf{T}^4$.

In principle, one can study the behavior of $Z_{n,m}(\gamma_1, ..., \gamma_n; \lambda\theta)$ for a fixed number m of handles. Let us examine the case $n = m = 0$ and $X = \mathbb{R}^4$. The partition function $Z_{0,0}$ is a sum over surfaces with the topology of S^2. Since σ has no boundary, the corresponding chain $\xi(\sigma)$ is actually a cycle, i.e.

$$\partial\xi(\sigma) = 0 \tag{5.1}$$

But $K(X = \mathbb{R}^4)$ is homologicaly trivial, and all closed chains are also exact [21]. Therefore

$$\xi(\sigma) = \partial\omega \tag{5.2}$$

for some $\omega \in C_{(1)}(K(X), \mathbb{Z})$. From (4.31), (4.32) and (5.2) one sees immediately that

$$I[\sigma] = 0. \tag{5.3}$$

Even though σ can self-intersect at many points, the intersection numbers add up to zero[†]. In the computation of $Z_{0,0}$, it does not matter if the θ-angle is zero or not. In other words

$$Z_{0,0}(\lambda, \theta) = Z_{0,0}(\lambda, 0) \tag{5.4}$$

Contrary to $Z_{0,0}$, the "n point" Green functions $Z_{n,0}$ depend on θ. For example, let us examine $Z_{1,0}$. The sum is now performed over the set $\Gamma_d(\gamma)$ of surfaces σ with boundary γ and no handles, in other words, surfaces with the topology of a disk. In contrast with (5.3), one can easily show that there are surfaces in $\Gamma_d(\gamma)$ that have self-intersection numbers different from zero.

Consider the following construction. Let σ_0 be a transversal surface with the topology of S^2. Suppose that σ_0 self-intersects at $2k$ points $v_i \in K(X)$. For each v_i there is a corresponding pair of points p_i and p_i' in the abstract cell complex σ_0. Let us associate a "charge" $\pm\frac{1}{2}$ to each point p_i and p_i' according if the intersection number at v_i is ± 1. From (5.3) it follows that the total "charge" is zero. Consider now a loop γ dividing σ_0 into disks σ_1 and σ_2 with $\partial\sigma_1 = \partial(\sigma_2) = \gamma$. Some pairs of points p_i, p_i' will be completely contained in σ_1 and some others will have one point in σ_1 and the other point in σ_2. (We assume that γ does not touch any intersection.) Let us call q_i the "charge" in σ_i $(i = 1, 2)$ due to the pairs that are not divided by γ, and q_{12} the remaining "charge". Then

$$q_1 + q_{12} + q_2 = 0. \tag{5.5}$$

[†]In particular, for transversal configurations the total number of intersection points has to be even.

The intersection number $I[\sigma_i]$ is obviously equal to q_i. Then, the contribution of σ_1 and σ_2 to $Z_1(\gamma;\theta)$ is given by

$$e^{iq_1\theta}e^{-\lambda A[\sigma_1]} + e^{iq_2\theta}e^{-\lambda A[\sigma_2]} \tag{5.6}$$

In particular, if $\theta = \pi$, q_1 is even and q_2 is odd, the contribution (5.6) reduces to

$$e^{-\lambda A[\sigma_1]} - e^{-\lambda A[\sigma_2]}. \tag{5.7}$$

From (5.5), one can see that q_1 and q_2 are integers such that $-k \leq q_1 + q_2 \leq k$. Therefore, $Z_1(\gamma;\lambda,\theta)$ does depend on θ.

Nothing much is known about the critical behavior of the model in the entire parameter space (λ,θ), except for $\theta = 0$. Unfortunately, for $\theta = 0$ the continuum limit is trivial. The sickness of the model at $\theta = 0$ is a consequence of the fact that the bare string tension has no zeros [7]. However, due to (5.6), the bare string tension

$$\tau(\lambda,\theta) = \lim_{\gamma\to\infty} \frac{1}{LM} \log Z_{1,0}(\gamma_{LM};\lambda,\theta).$$

can have a radically different behavior for $\theta \neq 0$. It is conceivable that, for $\theta = \pi$, there are critical points where $\tau(\lambda,\theta)$ does go to zero. The speculation of such a nontrivial continuum limit deserves further investigation.

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